# Form factor expansion of the row and diagonal correlation functions of the two dimensional Ising model

I. Lyberg * and B. M. McCoy †
*Department of Mathematics, State University of New York, Stony Brook, NY 11794-3840
†Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840
(Dated: February 7, 2008)

We derive and prove exponential and form factor expansions of the row correlation function $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and the diagonal correlation function $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ of the two dimensional Ising model.

YITP-SB-06-57

**Keywords:** Two dimensional Ising model, correlation functions, form factors.



## I. INTRODUCTION

The correlation functions $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ of the two dimensional Ising model with horizontal (vertical) interaction energies $E_1$ ($E_2$) can be written in many different ways which appear to be different but which in fact are equal. They were first expressed as determinants by Kaufman and Onsager [1]. Later Montroll, Potts and Ward [2] demonstrated that if an arbitrary path is drawn on the lattice connecting the point $(0,0)$ with the point $(M,N)$ then the correlation can be expressed as a determinant whose size in general is twice the length of the path. The correlations $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ can both be expressed as $N\times N$ Toeplitz determinants [1]–[3], and expressions of $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ as determinants of size $M$ and $M+1$ for $M\geq N$ were given by Yamada [4], [5]. Furthermore the correlations $\langle\sigma_{0,0}\sigma_{M,N}\rangle$ for all finite $M,\ N$ were expressed as determinants of Fredholm operators by Cheng and Wu [6].

The representations of the correlations as finite size determinants gives an efficient evaluation when the separation is small but to investigate the large separation behavior alternative representations are needed. The first such result is the limiting behavior for $T<T_c$

$$\begin{aligned} S_\infty = & \lim_{N\to\infty}\langle\sigma_{00}\sigma_{0N}\rangle = \lim_{N\to\infty}\langle\sigma_{00}\sigma_{NN}\rangle \\ = & \left\{1-(\sinh 2E_1/kT\sinh 2E_2/kT)^{-2}\right\}^{1/4}, \end{aligned} \tag{1}$$

which is most easily computed [2] by the use of Szegö's theorem [8],[9].

The first large separation expansion for both $T<T_c$ and $T>T_c$ beyond the limiting value (1) was given in 1966 by Wu [7] for $\langle\sigma_{00}\sigma_{0N}\rangle$ by applying a Wiener-Hopf procedure to the $N\times N$ Toeplitz determinant representation. Shortly thereafter Cheng and Wu [6] obtained the leading term of the large separation behavior of $\langle\sigma_{00}\sigma_{MN}\rangle$ by applying a Wiener-Hopf procedure to the Fredholm determinant representation. This derivation is formally valid only for $M\neq 0$, and even though it is expected that the result of [6] with $M$ formally set equal to zero should agree with the result of [7], there is no analytic derivation in the literature that for $T<T_c$ the two results are in fact equal (even though the equality has been verified to large orders in the low temperature expansion.)

The expansions of [7] and [6] may be considered as the first terms in a systematic expansion of the correlations. The expansion technique of [6] which starts from the Fredholm determinant representation was carried out to all orders by Wu, McCoy, Tracy and Barouch [10] in 1976 where it is found that the correlations can be written in the following exponential representation

$$\langle\sigma_{00}\sigma_{MN}\rangle_{T<T_c} = S_\infty \exp\sum_{n=1}^{\infty}F_{MN}^{(2n)} \quad \text{for } T<T_c \tag{2}$$

and as

$$\langle\sigma_{00}\sigma_{MN}\rangle_{T>T_c} = \widehat{S}_\infty \sum_{m=0}^{\infty}G_{MN}^{(2m+1)}\exp\sum_{n=1}^{\infty}\widehat{F}_{MN}^{(2n)} \quad \text{for } T>T_c \tag{3}$$

* e-mail ilyberg@math.sunysb.edu
† e-mail mccoy@max2.physics.sunysb.edu

where

$$\widehat{S}_{\infty} = \left\{1 - (\sinh 2E_1/kT \sinh 2E_2/kT)^2\right\}^{1/4}. \tag{4}$$

In [10] the expressions for $F^{(j)}_{MN}$, $\widehat{F}^{(j)}_{MN}$ and $G^{(j)}_{MN}$ are given as $2j$ fold multiple dimensional integrals.

The exponentials in (2) and (3) may be expanded to give what is called a form factor expansion

$$\langle\sigma_{00}\sigma_{MN}\rangle_{T<T_c} = S_{\infty}\sum_{n=0}^{\infty} f^{(2n)}_{MN} \quad \text{for} \quad T < T_c \tag{5}$$

and

$$\langle\sigma_{00}\sigma_{MN}\rangle_{T>T_c} = \widehat{S}_{\infty}\sum_{n=0}^{\infty} f^{(2n+1)}_{MN} \quad \text{for} \quad T > T_c. \tag{6}$$

The first few terms in this expansion were given in [10]. In the scaling limit $N \to \infty$, $T \to T_c$ with $N|T - T_c|$ fixed the full expansion was given by Nappi [11]. For fixed $N$ and $T < T_c$ the full expansion (5) was given by Palmer and Tracy [12]. Both of the cases $T < T_c$ and $T > T_c$ were treated by Nickel [14] [15]. An independent expansion was given by Yamada [13], and this is shown in [15] to agree with the results from the expansion of the exponential forms of [10].

The results for the exponential representation of the correlations [10] were obtained by extending to all orders the interative expansion of the Fredholm determinant representation [6]. However, as noted above, the result of [6] for $F^{(2)}_{M,N}$ when specialized to $M = 0$ "looks different" from the corresponding result for $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ obtained in [7]. Moreover the leading order large $N$ behavior of $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ is obtained [16] from the results for of [7] for $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and this result looks very different from the result of [10]. Therefore it must be the case that if the Wiener-Hopf procedure of Wu [7], which starts from the $N \times N$ Toeplitz determinant representation of $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$, is iterated to all orders we will obtain a representation of $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ which is different from that of [10]. The purpose of this paper is to derive these exponential expansions for both $T < T_c$ and $T > T_c$ and the form factor expansions which follow from them. The results for the case of the diagonal correlation have been previously presented by one of us (BM) in [17] with Boukraa, Hassani, Maillard, Orrick and Zenine. This paper is the proof, derivation and generalization of those results.

In sec. II we summarize the results of our calculations. In section III we calculate the exponential representation of the correlation functions $\langle\sigma_{00}\sigma_{0N}\rangle$ and $\langle\sigma_{00}\sigma_{NN}\rangle$ for $T < T_c$. In section IV we calculate the exponential representations for $T > T_c$. In section V we calculate the form factor expansions of $\langle\sigma_{00}\sigma_{0N}\rangle$ and $\langle\sigma_{00}\sigma_{NN}\rangle$ for $T < T_c$ and section VI we calculate the form factor expansions for $T > T_c$, We conclude in sec. VII with a brief discussion of our results.

## II. SUMMARY OF RESULTS

We let $D_N$ stand for $S_N = \langle\sigma_{00}\sigma_{0N}\rangle$ or $C_N = \langle\sigma_{00}\sigma_{NN}\rangle$. Then

$$D_N = \begin{cases} D^{(-)}_N & \text{for } T < T_c, \\ D^{(+)}_N & \text{for } T > T_c, \end{cases} \tag{7}$$

The representation of these correlations as an $N \times N$ Toeplitz determinant is [16]

$$D_N = \det \mathbf{A}_N \tag{8}$$

where

$$\mathbf{A}_N = \begin{pmatrix} a_0 & a_{-1} & \dots & a_{1-N} \\ a_1 & a_0 & \dots & a_{2-N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & a_{N-2} & \dots & a_0 \end{pmatrix} \tag{9}$$

and

$$a_n = \frac{1}{2\pi i}\oint_{|z|=1} \varphi(z) z^{-n-1}\, dz, \tag{10}$$

where the path of integration is counterclockwise. The function $\varphi(z)$ is

$$\varphi(z) = \left( \frac{(1-\alpha_1 z)(1-\alpha_2 z^{-1})}{(1-\alpha_1 z^{-1})(1-\alpha_2 z)} \right)^{1/2}. \tag{11}$$

For the diagonal correlation function $C_N$

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = (\sinh 2K_1 \sinh 2K_2)^{-1} \tag{12}$$

where $K_j = E_j/kT$. For the row correlation function $S_N$

$$\alpha_1 = e^{-2K_2} \tanh K_1 \quad \text{and} \quad \alpha_2 = e^{-2K_2} \coth K_1. \tag{13}$$

When $T < T_c$, then $\alpha_1 < \alpha_2 < 1$. In this case we write $\varphi$ in a factored form as

$$\varphi(z) = P(z)^{-1} Q(z^{-1})^{-1} \tag{14}$$

where

$$P(z) = ((1-\alpha_2 z)/(1-\alpha_1 z))^{1/2} \tag{15}$$

and

$$Q(z) = ((1-\alpha_1 z)/(1-\alpha_2 z))^{1/2} = 1/P(z) \tag{16}$$

We will prove in Sec. III that the correlation function $D_N^{(-)}$ has an exponential expansion

$$D_N^{(-)} = S_\infty \exp \sum_{n=1}^{\infty} F_N^{(2n)} \tag{17}$$

where

$$S_\infty = \left[ \frac{(1-\alpha_1^2)(1-\alpha_2^2)}{(1-\alpha_1\alpha_2)^2} \right]^{1/4} \tag{18}$$

which for both the diagonal (12) and row (13) specializes to (1), and

$$F_N^{(2n)} = \frac{(-1)^{n+1}}{n(2\pi)^{2n}} \lim_{\epsilon \to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \prod_{j=1}^{2n} \frac{z_j^N}{1 - z_j z_{j+1}} \prod_{k=1}^{n} P(z_{2k}) P(z_{2k}^{-1}) Q(z_{2k-1}) Q(z_{2k-1}^{-1}) \tag{19}$$

and $z_{2n+1} = z_1$. This agrees with the result given in ref. [17] for the diagonal correlation function $C_N^{(-)}$.

In Sec. V we prove that $D_N^{(-)}$ has the form factor expansion

$$D_N^{(-)} = S_\infty \sum_{n=0}^{\infty} f_N^{(2n)} \tag{20}$$

where $f_N^{(0)} = 1$ and

$$\begin{aligned} f_N^{(2n)} = \quad & \frac{1}{(n!)^2 (2\pi)^{2n}} \lim_{\epsilon \to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^N \prod_{k=1}^{n} P(z_{2k}) P(z_{2k}^{-1}) Q(z_{2k-1}) Q(z_{2k-1}^{-1}) \\ & \prod_{l=1}^{n} \prod_{m=1}^{n} (1 - z_{2l-1} z_{2m})^{-2} \prod_{1 \le p < q \le n} (z_{2p-1} - z_{2q-1})^2 (z_{2p} - z_{2q})^2. \end{aligned} \tag{21}$$

This agrees with the result given in ref. [17] for the diagonal correlation function $C_N^{(-)}$.

For $T > T_c$, we consider a new function $\varphi_1(z)$ such that

$$\varphi_1(z) = \varphi(z) z = \left( \frac{(1-\alpha_1 z)(1-\alpha_2^{-1} z)}{(1-\alpha_1 z^{-1})(1-\alpha_2^{-1} z^{-1})} \right)^{1/2} \tag{22}$$

which we write in factored form as

$$\varphi_1(z) = \widehat{P}(z)^{-1}\widehat{Q}(z^{-1})^{-1} \tag{23}$$

with

$$\widehat{P}(z) = ((1-\alpha_1 z)(1-\alpha_2^{-1}z))^{-1/2} \tag{24}$$

and

$$\widehat{Q}(z) = ((1-\alpha_1 z)(1-\alpha_2^{-1}z))^{1/2} = 1/\widehat{P}(z). \tag{25}$$

$\widehat{P}(z)$ and $\widehat{Q}(z)$ are analytic and non-zero for $|z|<1$.

We prove in Sec. IV that the correlation function $D_N^{(+)}$ has an exponential expansion

$$D_N^{(+)} = -\widehat{S}_\infty \sum_{m=0}^{\infty} G_N^{(2m+1)} \exp \sum_{n=1}^{\infty} \widehat{F}_{N+1}^{(2n)} \tag{26}$$

where

$$\widehat{S}_\infty = \left[(1-\alpha_1^2)(1-\alpha_2^{-2})(1-\alpha_1\alpha_2^{-1})^2\right]^{1/4} \tag{27}$$

which for both the diagonal (12) and row (13) correlations specializes to (4) and $\widehat{F}_N^{(2n)}$ is defined as in (19), but with $P$ and $Q$ replaced by $\widehat{P}$ and $\widehat{Q}$. Thus we find from (19) that $\widehat{F}_N^{(2n)}$ is

$$\widehat{F}_N^{(2n)} = \frac{(-1)^{n+1}}{n(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \prod_{j=1}^{2n} \frac{z_j^N}{1-z_j z_{j+1}} \prod_{k=1}^{n} \widehat{P}(z_{2k})\widehat{P}(z_{2k}^{-1})\widehat{Q}(z_{2k-1})\widehat{Q}(z_{2k-1}^{-1}) \tag{28}$$

and $G_N^{(2n+1)}$ is given by

$$\begin{aligned} G_N^{(2n+1)} = \frac{1}{(2\pi i)^{2n+1}} \quad & \lim_{\epsilon\to 0} \prod_{i=1}^{2n+1} \oint_{|z_i|=1-\epsilon} dz_i \ z_i^{N+1} \frac{1}{z_1 z_{2n+1}} \prod_{k=1}^{2n} \frac{1}{1-z_k z_{k+1}} \\ & \prod_{l=1}^{n+1} \widehat{P}(z_{2l-1})\widehat{P}(z_{2l-1}^{-1}) \prod_{m=1}^{n} \widehat{Q}(z_{2m})\widehat{Q}(z_{2m}^{-1}). \end{aligned} \tag{29}$$

Equations (28) and (29) agree with the results given in ref. [17]. Note that for the diagonal correlation function $C_N^{(+)} = \langle\sigma_{00}\sigma_{NN}\rangle$ (12) implies that

$$\widehat{F}_N^{(2n)} = F_N^{(2n)}. \tag{30}$$

In Sec. VI we prove that $D_N^{(+)}$ has the form factor expansion

$$D_N^{(+)} = -\widehat{S}_\infty \sum_{n=0}^{\infty} f_N^{(2n+1)} \tag{31}$$

where

$$\begin{aligned} f_N^{(2n+1)} = -\frac{i}{n!(n+1)!(2\pi)^{2n+1}} & \lim_{\epsilon\to 0} \prod_{i=1}^{2n+1} \oint_{|z_i|=1-\epsilon} dz_i \ z_i^N \prod_{l=1}^{n+1} \widehat{P}(z_{2l-1})\widehat{P}(z_{2l-1}^{-1}) z_{2l-1}^{-1} \prod_{m=1}^{n} \widehat{Q}(z_{2m})\widehat{Q}(z_{2m}^{-1}) z_{2m} \\ & \prod_{p=1}^{n+1}\prod_{q=1}^{n} \frac{1}{(1-z_{2p-1}z_{2q})^2} \prod_{1\le j<k\le n+1} (z_{2j-1}-z_{2k-1})^2 \prod_{1\le r<s\le n} (z_{2r}-z_{2s})^2. \end{aligned} \tag{32}$$

Equation (32) agrees with result given in ref. [17] for the diagonal correlation function $C_N^{(+)}$.

The proofs of these results are not restricted to the Ising case where the generating function is given by (11) but with a suitable replacement for the factors $S_\infty$ and $\widehat{S}_\infty$ are valid in more general cases, for example the XY model in a magnetic field [18]-[20]. The results (17)-(21) for $T<T_c$ are valid for any generating function $\varphi(z)$ where $\log\varphi(z)$ is analytic and periodic on $|z|=1$ and $P(z)=1/Q(z)$ The results (26)-(29) and (26)-(32) for $T>T_c$ are similarly valid for any generating function for which $\log z\varphi(z)$ is analytic and periodic on the unit circle $|z|=1$ and $\widehat{P}(z)=1/\widehat{Q}(z)$.

## III. THE EXPONENTIAL EXPANSION FOR $T < T_c$

In this section, we will use the theory of Wiener-Hopf sum equations to prove that the functions $F_N^{(2n)}$ which appear in equation (17) are given by (19).

When $T < T_c$, then $\alpha_1 < \alpha_2 < 1$ and therefore $P(z)$ and $Q(z)$ are analytic and non-zero for $|z| < 1$. Furthermore the index of $\varphi$ is

$$\mathrm{Ind}\varphi = \log \varphi(e^{2\pi i}) - \log \varphi(1) = 0 \tag{33}$$

It follows from (33) that we may use Szegö's theorem to find

$$\lim_{N\to\infty} D_N^{(-)} = S_\infty \tag{34}$$

with $S_\infty$ given (18) which reduces to (1) for both the diagonal and the row correlation functions. Therefore we may write

$$D_N^{(-)} = S_\infty \prod_{n=N}^{\infty} D_n^{(-)}/D_{n+1}^{(-)} \tag{35}$$

### A. Computation of the ratio $D_N^{(-)}/D_{N+1}^{(-)}$

The ratio $D_N^{(-)}/D_{N+1}^{(-)}$ is given by

$$D_N^{(-)}/D_{N+1}^{(-)} = x_0^{(N)} \tag{36}$$

where $\mathbf{x}^{(N)} = (x_0, x_1, ..., x_N)$ satisfies

$$\mathbf{A}_{N+1}\mathbf{x}^{(N)} = \mathbf{d}^{(N)} \tag{37}$$

and $d_i^{(N)} = \delta_{i0}$. We indicate that the vector $\mathbf{x}^{(N)}$ has $N+1$ entries by writing $x_0^{(N)}$.

We will calculate $x_0^{(N)}$ by iterating the procedure given by Wu in section 3 of reference [7].

**Lemma 1** There are functions $\phi_N^{(2n)}$ such that

$$x_0^{(N)} = 1 + \sum_{n=1}^{\infty} \phi_N^{(2n)} \tag{38}$$

where

$$\phi_N^{(2n)} = \frac{(-1)^{n+1}}{(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^{N+1} \frac{1}{z_1 z_{2n}} \prod_{k=1}^{n} Q(z_{2k-1})Q(z_{2k-1}^{-1})P(z_{2k})P(z_{2k}^{-1}) \prod_{l=1}^{2n-1} \frac{1}{1 - z_l z_{l+1}}. \tag{39}$$

**Proof** Let $h(\xi)$ be a function defined on the unit circle $|\xi| = 1$, and let $h(\xi)$ have the Laurent expansion

$$h(\xi) = \sum_{n=-\infty}^{\infty} h_n \xi^n. \tag{40}$$

From this we define

$$[h(\xi)]_+ = \sum_{n=0}^{\infty} h_n \xi^n, \qquad [h(\xi)]_- = \sum_{n=-\infty}^{-1} h_n \xi^n, \qquad \text{and} \qquad [h(\xi)]'_+ = \sum_{n=1}^{\infty} h_n \xi^n. \tag{41}$$

From equations (41) it follows that

$$[h(\xi^{-1})]_- = [h(\xi)]'_+. \tag{42}$$

Equations (41) have the integral representations

$$[h(\xi)]_+ = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{|\xi'|=1+\epsilon} d\xi' \frac{h(\xi')}{\xi'-\xi}, \tag{43}$$

$$[h(\xi)]_- = \frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{|\xi'|=1-\epsilon} d\xi' \frac{h(\xi')}{\xi-\xi'}, \tag{44}$$

and

$$\begin{aligned} [h(\xi)]'_+ = & \quad [h(\xi)]_+ - \frac{1}{2\pi i}\oint_{|\xi|=1} d\xi \; \frac{h(\xi)}{\xi} \\ = & \quad \frac{1}{2\pi i}\,\xi\lim_{\epsilon\to 0}\oint_{|\xi'|=1+\epsilon} d\xi' \; \frac{h(\xi')}{\xi'(\xi'-\xi)}. \end{aligned} \tag{45}$$

We define

$$X_N(\xi) = \sum_{n=0}^{N-1} x_n^{(N)}\xi^n \tag{46}$$

It has been proven by Wu [7] that the ratio (36) is given by

$$x_0^{(N)} = X_N(0) \tag{47}$$

where $X_N(\xi)$ is a function determined by equations (2.19a)-(2.20b) of reference [7] (with $Y(\xi)=1$). These equations are

$$X_N(\xi) = P(\xi)\left\{[Q(\xi^{-1})]_+ + [Q(\xi^{-1})U_N(\xi)\xi^N]_+\right\} \qquad (2.19\text{a}), \tag{48}$$

$$V_N(\xi^{-1}) = -(Q(\xi^{-1}))^{-1}\left\{[Q(\xi^{-1})]_- + [Q(\xi^{-1})U_N(\xi)\xi^N]_-\right\} \qquad (2.20\text{a}), \tag{49}$$

$$X_N(\xi^{-1})\xi^N = Q(\xi)\left\{[P(\xi^{-1})\xi^N]_+ + [P(\xi^{-1})V(\xi)\xi^N]_+\right\} \qquad (2.19\text{b}), \tag{50}$$

and

$$U_N(\xi^{-1}) = -(P(\xi^{-1}))^{-1}\left\{[P(\xi^{-1})\xi^N]_- + [P(\xi^{-1})V_N(\xi)\xi^N]_-\right\} \qquad (2.20\text{b}). \tag{51}$$

For our purposes we use equations (16), (42) and the equality $[Q(\xi^{-1})]_+ = 1$ to rewrite equations (48), (49) and (51) as

$$X_N(\xi) = P(\xi)\left\{1 + [Q(\xi^{-1})U_N(\xi)\xi^N]_+\right\}, \tag{52}$$

$$V_N(\xi^{-1}) = -P(\xi^{-1})\left\{[Q(\xi^{-1})]_- + [Q(\xi^{-1})U_N(\xi)\xi^N]_-\right\}, \tag{53}$$

and

$$U_N(\xi) = -Q(\xi)\left\{[P(\xi)\xi^{-N}]'_+ + [P(\xi)V_N(\xi^{-1})\xi^{-N}]'_+\right\} \tag{54}$$

We define $V_N^{(1)}(\xi^{-1})$ by replacing $U_N(\xi)$ by 0 in equation (53). Thus

$$V_N^{(1)}(\xi^{-1}) = -P(\xi^{-1})[Q(\xi^{-1})]_-. \tag{55}$$

We note from equation (16) that $Q(0) = 1$. Thus because $Q(\xi^{-1})$ is analytic for $|\xi| > 1$, we have

$$[Q(\xi^{-1})]_- = Q(\xi^{-1}) - Q(0) = Q(\xi^{-1}) - 1. \tag{56}$$

Therefore it follows from equations (16) and (56) that

$$-P(\xi^{-1})[Q(\xi^{-1})]_- = P(\xi^{-1}) - 1, \tag{57}$$

and thus equation (55) becomes

$$V^{(1)}(\xi^{-1}) = P(\xi^{-1}) - 1. \tag{58}$$

We define $U^{(1)}(\xi)$ by replacing $V_N(\xi^{-1})$ in (54) $V_N^{(1)}(\xi^{-1})$ as given by equation (58). Thus we find

$$U_N^{(1)}(\xi) = -Q(\xi)[P(\xi^{-1})P(\xi)\xi^{-N}]'_+ \tag{59}$$

It follows from equation (52) that $X_N^{(1)}(\xi)$ is given by

$$\begin{aligned} X_N^{(1)}(\xi) &= P(\xi)\Big\{1 - \big[Q(\xi^{-1})Q(\xi)[P(\xi^{-1})P(\xi)\xi^{-N}]'_+\xi^N\big]_+\Big\} \\ &= P(\xi)\left\{1 - \frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{|\xi'|=1+\epsilon} d\xi' \frac{\xi'^N}{\xi' - \xi} Q(\xi^{-1})Q(\xi)[P(\xi^{-1})P(\xi)\xi^{-N}]'_+\right\}. \end{aligned} \tag{60}$$

Letting $\xi = 0$ in equation (60), and using $P(0) = 1$, and writing $X^{(1)}(0) = 1 + \phi_N^{(2)}$ we obtain

$$\begin{aligned} \phi_N^{(2)} &= -\frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\ Q(\xi^{-1})Q(\xi)[P(\xi^{-1})P(\xi)\xi^{-N}]'_+\xi^{N-1} \\ &= -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{|\xi|=1} d\xi_1\ Q(\xi_1^{-1})Q(\xi_1)\frac{1}{2\pi i}\xi_1^N \oint_{|\xi_2|=1+\epsilon} d\xi_2 \frac{1}{\xi_2}\frac{1}{\xi_2 - \xi_1} P(\xi_2^{-1})P(\xi_2)\xi_2^{-N}. \end{aligned} \tag{61}$$

Thus, if we set

$$\xi_{2k+1} = z_{2k+1}, \qquad \xi_{2k} = z_{2k}^{-1} \tag{62}$$

we obtain $\phi_N^{(2)}$ as given by equation (39).

We now calculate $V_N^{(2)}(\xi^{-1})$ by using equation (59) in equation (53):

$$\begin{aligned} V_N^{(2)}(\xi^{-1}) &= -P(\xi^{-1})\Big\{[Q(\xi^{-1})]_- + [Q(\xi^{-1})U_N^{(1)}(\xi)\xi^N]\Big\} \\ &= -P(\xi^{-1})[Q(\xi^{-1})]_- + P(\xi^{-1})\big[Q(\xi^{-1})Q(\xi)\xi^N[P(\xi^{-1})P(\xi)\xi^{-N}]'_+\big]_-. \end{aligned} \tag{63}$$

Next, we calculate $U_N^{(2)}(\xi)$ by using equation (63) in equation (54):

$$\begin{aligned} U_N^{(2)}(\xi) &= -(P(\xi))^{-1}\Big\{[P(\xi)\xi^{-N}]'_+ + [P(\xi)V_N^{(2)}(\xi^{-1})\xi^{-N}]'_+\Big\} \\ &= -Q(\xi)[P(\xi)P(\xi^{-1})\xi^{-N}]'_+ - Q(\xi)\Big[P(\xi)P(\xi^{-1})\xi^{-N}\big[Q(\xi)Q(\xi^{-1})\xi^N[P(\xi)P(\xi^{-1})\xi^{-N}]'_+\big]_-\Big]'_+. \end{aligned} \tag{64}$$

We will now calculate $X_N^{(2)}(\xi)$ from (52) as

$$\begin{aligned} X_N^{(2)}(\xi) &= P(\xi)\{1 + [Q(\xi^{-1})U_N^{(2)}(\xi)\xi^N]_+\} \\ &= P(\xi) - P(\xi)\big[Q(\xi^{-1})Q(\xi)[P(\xi)P(\xi^{-1})\xi^{-N}]'_+\xi^N\big]_+ \\ &\quad -P(\xi)\Big[Q(\xi^{-1})Q(\xi)\xi^N\Big[P(\xi)P(\xi^{-1})\xi^{-N}\big[Q(\xi)Q(\xi^{-1})\xi^N[P(\xi)P(\xi^{-1})\xi^{-N}]'_+\big]_-\Big]'_+\Big]_+. \end{aligned} \tag{65}$$

Letting $\xi = 0$ in equation (65), we obtain $X_N^{(2)}(0) = 1 + \phi_N^{(2)} + \phi_N^{(4)}$:

$$\begin{aligned}\phi_N^{(4)} &= -\frac{1}{2\pi i}\oint_{|\xi_1|=1} d\xi\; Q(\xi^{-1})Q(\xi)\Big[P(\xi^{-1})P(\xi)\xi^{-N}\big[Q(\xi^{-1})Q(\xi)\xi^N[P(\xi^{-1})P(\xi)\xi^{-N}]'_+\big]_-\Big]'_+\xi^{N-1} \\ &= -\frac{1}{(2\pi i)^4}\lim_{\epsilon\to 0}\oint_{|\xi|=1} d\xi_1\; \xi_1^N Q(\xi_1^{-1})Q(\xi_1)\oint_{|\xi_2|=1+\epsilon} d\xi_2\;\frac{1}{\xi_2-\xi_1}\xi_2^{-N-1}P(\xi_2^{-1})P(\xi_2) \\ &\oint_{|\xi_3|=1} d\xi_3\;\frac{1}{\xi_3-\xi_2}\xi_3^{N+1}Q(\xi_3^{-1})Q(\xi_3)\oint_{|\xi_4|=1+\epsilon} d\xi_4\;\frac{1}{\xi_4-\xi_3}\xi_4^{-N-1}P(\xi_4^{-1})P(\xi_4).\end{aligned} \tag{66}$$

Using the change of variables (62) we obtain an equation agreeing with equation (39).

In general, we iteratively define (from equation 53)

$$V_N^{(n+1)}(\xi^{-1}) := -P(\xi^{-1})\left\{[Q(\xi^{-1})]_- + [Q(\xi^{-1})U_N^{(n)}(\xi)\xi^N]_-\right\}. \tag{67}$$

It then follows from equation (54) that

$$U_N^{(n)}(\xi) - U_N^{(n-1)}(\xi) = -Q(\xi^{-1})\left[P(\xi)P(\xi^{-1})\xi^{-N}\left[Q(\xi)Q(\xi^{-1})\xi^N\left[P(\xi)P(\xi^{-1})\xi^{-N}[Q(\xi)Q(\xi^{-1})\xi^N...]_-\right]'_+\right]_-\right]'_+, \tag{68}$$

where there are $2n-1$ brackets. It now follows from equations (47) and and (52) that $\phi_N^{(2k)}$ is

$$\begin{aligned}\phi_N^{(2k)} = -\frac{1}{2\pi i}\oint_{|\xi|=1} d\xi\; \xi^{N-1}Q(\xi)Q(\xi^{-1})\Big[P(\xi)P(\xi^{-1})\xi^{-N}\Big[Q(\xi)Q(\xi^{-1})\xi^N \\ \big[P(\xi)P(\xi^{-1})\xi^{-N}[Q(\xi)Q(\xi^{-1})\xi^N...]_-\big]'_+\Big]_-\Big]'_+,\end{aligned} \tag{69}$$

where there are $2k-1$ brackets. By use of (62), one obtains equation (39). This ends the proof of the lemma.

### B. Exponentiation

To complete the proof of the exponential form (17) we need to use (35), (36) and (38) to compute $F_N^{(2n)}$ as given in (19). We begin by defining a function

$$\tilde{F}_N^{(2n)} := \frac{(-1)^{n+1}}{n(2\pi)^{2n}}\lim_{\epsilon\to 0}\prod_{i=1}^{2n}\oint_{|z_i|=1-\epsilon} dz_i\prod_{j=1}^{2n}\frac{z_j^N}{1-z_jz_{j+1}}\prod_{l=1}^{n}Q(z_{2l-1})Q(z_{2l-1}^{-1})P(z_{2l})P(z_{2l}^{-1})\left(1-\prod_{k=1}^{2n}z_k\right) \tag{70}$$

(We define $\tilde{F}_N^{(0)} = 0$). Clearly

$$F_N^{(2n)} = \sum_{k=N}^{\infty}\tilde{F}_k^{(2n)}. \tag{71}$$

Let $\phi_N^{(2n)}$ be given by equation (39) when $n \geq 1$ and let $\phi_N^{(0)} = 1$. We define the functions

$$\phi(\lambda) := \sum_{n=0}^{\infty}\phi_N^{(2n)}\lambda^n \tag{72}$$

and

$$\tilde{F}(\lambda) := \sum_{n=0}^{\infty}\tilde{F}_N^{(2n)}\lambda^n. \tag{73}$$

Clearly $\phi(0) = 1$ and $F(0) = 0$. We would like to show that

$$\phi(\lambda) = e^{\tilde{F}(\lambda)} \tag{74}$$

It follows as a special case of (74) with $\lambda = 1$ that

$$X_N(0) = \exp \sum_{k=1}^{\infty} \tilde{F}_N^{(2k)}, \tag{75}$$

and hence it follows from equations (35) and (71) that

$$C_N \quad = (1-t)^{1/4} \exp \sum_{k=N}^{\infty} \sum_{n=1}^{\infty} \tilde{F}_k^{(2n)} = (1-t)^{1/4} \exp \sum_{n=1}^{\infty} \sum_{k=N}^{\infty} \tilde{F}_k^{(2n)} = (1-t)^{1/4} \exp \sum_{n=1}^{\infty} F_N^{(2n)} \tag{76}$$

This proves equation (19). It remains to show that equation (74) holds. Since $\phi(0) = 1$ and $F(0) = 0$, equation (74) is equivalent to the equation

$$\phi'(\lambda) = e^{\tilde{F}(x)} \tilde{F}'(\lambda) \tag{77}$$

It follows from equations (72), (73) and (77) that equation (74) is equivalent to the following equation:

**Lemma 2**

$$n\phi_N^{(2n)} = \sum_{l=1}^{n} l \tilde{F}_N^{(2l)} \phi_N^{(2n-2l)} \tag{78}$$

**Proof** It follows from (39) that the left hand side of (78) is

$$n\phi_N^{(2n)} = n(-1)^{n+1} \frac{1}{(2\pi)^{2n}} \lim_{\epsilon \to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^{N+1} \frac{1}{z_{2n} z_1} \prod_{j=1}^{n} P(z_{2j}) P(z_{2j}^{-1}) Q(z_{2j-1}) Q(z_{2j-1}^{-1}) \prod_{k=1}^{2n-1} \frac{1}{1 - z_k z_{k+1}}, \tag{79}$$

and the right hand side is

$$\begin{aligned} \sum_{l=1}^{n} l \tilde{F}_N^{(2l)} \phi_N^{(2n-2l)} \quad &= (-1)^n \frac{1}{(2\pi)^{2n}} \lim_{\epsilon \to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \prod_{j=1}^{2n} \frac{z_j^N}{1 - z_j z_{j+1}} \prod_{q=1}^{n} P(z_{2q}) P(z_{2q}^{-1}) Q(z_{2q-1}) Q(z_{2q-1}^{-1}) \\ &\left\{ \sum_{l=1}^{n-1} \frac{1}{1 - z_1 z_{2l}} \left( 1 - \prod_{k=1}^{2l} z_k \right) (1 - z_{2l} z_{2l+1})(1 - z_{2n} z_1) \prod_{m=2l+2}^{2n-1} z_m - \left( 1 - \prod_{p=1}^{2n} z_p \right) \right\}, \end{aligned} \tag{80}$$

where the product $\prod_{m=2l+2}^{2n-1} z_m$ is such that it equals 1 when $l = n-1$. Note that the product $\prod_{j=1}^{2n}$ is symmetric both in even and in odd variables separately. Hence $1 - \prod_{k=1}^{2n} z_k$ can be rewritten (under integration) as

$$1 - \prod_{k=1}^{2n} z_k \equiv (1 - z_1 z_{2n}) \left( 1 + \sum_{q=1}^{n-1} \prod_{r=2}^{2q+1} z_r \right). \tag{81}$$

Next, note that the factor $\frac{1}{1 - z_1 z_{2l}} (1 - z_{2l} z_{2l+1})(1 - z_{2n} z_1) \prod_{m=2l+2}^{2n-1} z_m$ does not involve any of the variables $\{z_i\}_{i=2}^{2l-1}$. Hence the product $1 - \prod_{k=1}^{2l} z_k$ can be rewritten as

$$1 - \prod_{k=1}^{2l} z_k \equiv (1 - z_1 z_{2l}) \left( 1 + \sum_{q=1}^{l-1} \prod_{r=2}^{2q+1} z_r \right). \tag{82}$$

Then the relevant factor of the integrand of the right hand side of equation (80) becomes

$$\begin{aligned} &(1 - z_{2n} z_1) \left\{ \sum_{l=1}^{n-1} (1 - z_{2l} z_{2l+1}) \left( 1 + \sum_{q=1}^{l-1} \prod_{r=2}^{2q+1} z_r \right) \prod_{m=2l+2}^{2n-1} z_m - \left( 1 + \sum_{q=1}^{n-1} \prod_{r=2}^{2q+1} z_r \right) \right\} \\ &= (1 - z_{2n} z_1) \left\{ \sum_{l=1}^{n-1} \left( 1 + \sum_{q=1}^{l-1} \prod_{r=2}^{2q+1} z_r \right) \left( \prod_{m=2l+2}^{2n-1} z_m - \prod_{m=2l}^{2n-1} z_m \right) - \left( 1 + \sum_{q=1}^{n-1} \prod_{r=2}^{2q+1} z_r \right) \right\}. \end{aligned} \tag{83}$$

After expansion of the first summand the right hand side of (83) becomes

$$(1-z_{2n}z_1)\left\{\sum_{l=1}^{n-1}\prod_{m=2l+2}^{2n-1} z_m - \sum_{l=1}^{n-1}\prod_{r=2}^{2n-1} z_r - \left(1+\sum_{q=1}^{n-1}\prod_{r=2}^{2q+1} z_r\right)\right\} \tag{84}$$

under integration. After summation (84) becomes

$$-\,n(1-z_{2n}z_1)\prod_{r=2}^{2n-1} z_r, \tag{85}$$

which completes the proof. The proof of lemma 2 concludes the proof of equation (19).

## IV. THE EXPONENTIAL EXPANSION FOR $T > T_c$

In this section, we will prove that the functions $\widehat{F}_N^{(2n)}$ and $G_N^{(2n+1)}$ in (26) are given by (28) and (29). We will follow the procedure of section 2 of Wu [7]. When $T > T_c$, then $\alpha_1 < 1 < \alpha_2$ and $\varphi_1(z)$ has index 0. We define

$$b_n := \frac{1}{2\pi i}\oint_{|z|=1} \varphi_1(z) z^{-n-1}\, dz = a_{n-1} \tag{86}$$

We further define

$$\mathbf{B}_{N+1} := \begin{pmatrix} b_0 & b_{-1} & \dots & b_{-N} \\ b_1 & b_0 & \dots & b_{1-N} \\ \vdots & \vdots & \ddots & \vdots \\ b_N & b_{N-1} & \dots & b_0 \end{pmatrix} \tag{87}$$

and

$$\widehat{D}_{N+1} := \det \mathbf{B}_{N+1} \tag{88}$$

We note that if we remove the first row and the last column from $\widehat{D}_{N+1}$ and use (86) we obtain $D_N$ as defined by (8). Therefore we may write

$$D_N^{(+)} = \frac{D_N^{(+)}}{\widehat{D}_{N+1}}\widehat{D}_{N+1} = (-1)^N x_N^{(N)} \widehat{D}_{N+1}, \tag{89}$$

where the ratio $D_N/\widehat{D}_{N+1}$ is given as

$$\frac{D_N^{(+)}(t)}{\widehat{D}_{N+1}(t)} = (-1)^N x_N^{(N)} \tag{90}$$

and $\mathbf{x}^{(N)} = (x_0, x_1, ..., x_N)$ satisfies

$$\mathbf{B}_{N+1}\mathbf{x}^{(N)} = \mathbf{d}^{(N)} \tag{91}$$

and $d_i^{(N)} = \delta_{i0}$. We indicate that the vector $\mathbf{x}^{(N)}$ has $N+1$ entries by writing $x_N^{(N)}$. Since $\varphi_1(z)$ has index 0, it follows from Szegö's theorem that

$$\lim_{N\to\infty} (-1)^N \widehat{D}_N = \widehat{S}_\infty \tag{92}$$

where $\widehat{S}_\infty$ is given by (27). Thus, exactly as for $T < T_c$,

$$(-1)^{N+1}\widehat{D}_{N+1} = \widehat{S}_\infty \prod_{n=N+1}^{\infty} \frac{\widehat{D}_n}{\widehat{D}_{n+1}}. \tag{93}$$

Furthermore the ratio $\widehat{D}_n/\widehat{D}_{n+1}$ and the product

$$\prod_{n=N+1}^{\infty} \frac{\widehat{D}_n}{\widehat{D}_{n+1}} \tag{94}$$

may be treated exactly as in the case $T < T_c$ if we replace $P$ and $Q$ by $\widehat{P}$ and $\widehat{Q}$. Thus we find

$$(-1)^{N+1}\widehat{D}_{N+1} = \widehat{S}_\infty \exp \sum_{n=1}^{\infty} \widehat{F}_{N+1}^{(2n)}, \tag{95}$$

and hence we have

$$D_N^{(+)}(t) = -\widehat{S}_\infty x_N^{(N)} \exp \sum_{n=1}^{\infty} \widehat{F}_{N+1}^{(2n)}, \tag{96}$$

where we note that when $\alpha_1 = 0$, equation (30) holds.

It remains to calculate $x_N^{(N)}$. We will find $x_N^{(N)}$ by iterating the procedure of section 2 of Wu [7]. We define

$$X_N(\xi) = \sum_{n=0}^{N} x_n^{(N)} \xi^n, \tag{97}$$

and thus

$$x_N^{(N)} = \lim_{\xi \to 0} X(\xi^{-1})\xi^N \tag{98}$$

where $X_N(\xi)$ is again defined by (48) to (51) with $P(\xi)$ and $Q(\xi)$ replaced by $\widehat{P}(\xi)$ and $\widehat{Q}(\xi)$. For convenience we rewrite (49), replacing $\xi$ with $\xi^{-1}$ as

$$\begin{aligned} V_N(\xi) &= -\widehat{P}(\xi)\left\{[\widehat{Q}(\xi)]'_+ + [\widehat{Q}(\xi)U_N(\xi^{-1})\xi^{-N}]'_+\right\} \\ &= \widehat{P}(\xi) - 1 - \widehat{P}(\xi)[\widehat{Q}(\xi)U_N(\xi^{-1})\xi^{-N}]'_+. \end{aligned} \tag{99}$$

To obtain the first approximation $x_N^{(N)(1)}$ we replace $U(\xi)$ by 0 in (99), and write

$$V_N^{(1)}(\xi) = \widehat{P}(\xi) - 1. \tag{100}$$

We use this in (50) to give

$$X_N^{(1)}(\xi^{-1})\xi^N = \widehat{Q}(\xi)[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N]_+. \tag{101}$$

Thus letting $\xi$ approach 0 and using (101) we obtain the first approximation $x_N^(N,1)$, which we denote as $G_N^{(1)}$:

$$G_N^{(1)} = x_N^{(N)(1)} = \frac{1}{2\pi i}\oint_{|\xi|=1} \widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^{N-1}d\xi. \tag{102}$$

We now compute the second approximation by using (100) in (51) to obtain

$$U^{(2)}(\xi^{-1}) = -\widehat{Q}(\xi^{-1})[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N]_-. \tag{103}$$

We use (103) in (99) to find

$$V_N^{(2)}(\xi) = \widehat{P}(\xi) - 1 + \widehat{P}(\xi)\left[\widehat{Q}(\xi)\widehat{Q}(\xi^{-1})\xi^{-N}[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N]_-\right]'_+. \tag{104}$$

Using this in (50) we obtain

$$X^{(2)}(\xi^{-1})\xi^N = \widehat{Q}(\xi)\Big\{[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N]_+ + \Big[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N\left[\widehat{Q}(\xi^{-1})\widehat{Q}(\xi)\xi^{-N}[\widehat{P}(\xi^{-1})\widehat{P}(\xi)\xi^N]_-\right]'_+\Big]_+\Big\}. \tag{105}$$

Letting $\xi = 0$ in (105), we see that

$$x_N^{(N)(3)} = G_N^{(1)} + G_N^{(3)}, \tag{106}$$

where

$$G_N^{(3)} = \frac{1}{(2\pi i)^3} \quad \lim_{\epsilon\to 0} \oint_{|z_1|=1} dz_1 \; z_1^N \widehat{P}(z_1)\widehat{P}(z_1^{-1}) \oint_{|z_2|=1-\epsilon} dz_2 \; \frac{z_2^{N+1}}{1-z_1 z_2} \widehat{Q}(z_2)\widehat{Q}(z_2^{-1})$$
$$\oint_{|z_3|=1} dz_3 \; \frac{z_3^N}{1-z_2 z_3} \widehat{P}(z_3)\widehat{P}(z_3^{-1}). \tag{107}$$

Continuing in the same way we may find

$$x_N^{(N)(2n+1)} = \sum_{k=0}^{n} G_N^{(2k+1)}, \tag{108}$$

and thus

$$D_N^{(+)}(t) = -(1-t)^{1/4} \sum_{n=0}^{\infty} G_N^{(2n+1)} \exp \sum_{m=1}^{\infty} \widehat{F}_{N+1}^{(2m)} \tag{109}$$

where $\widehat{F}_N^{(2n)}$ is defined in (28) and $G_N^{(2n+1)}$ is defined in (29).

If we note that the $G_N^{(2n+1)}$ is the negative of the $G_N^{(2n+1)}$ of [17] and set $\alpha_1 = 0$ we have proven (6) of [17] with $G_N^{(2n+1)}$ given by (34) of [17].

## V. THE FORM FACTOR EXPANSION FOR $T < T_c$

We have showed in section III that the correlation function $D_N^{(-)}$ can be written in an exponential form given by (17) and (19). In this section we will show that $D_N^{(-)}$ can be written as a form factor expansion given by equations (20) and (21).

We wish to rewrite (17) as a form factor expansion and use an argument similar to that made by Nappi [11] to find the functions $f_N^{(2n)}$. To do this, we denote by a partition $\pi$ of the number $n$ a set of pairs $\pi = \{(n_i, m_i)\}_{i=1}^{\nu(\pi)}$ such that $n_i \neq n_j$ if $i \neq j$ and

$$\sum_{i=1}^{\nu(\pi)} n_i m_i = n. \tag{110}$$

We define $\mathcal{P}(n)$ to be the set of all such partitions. For instance, the partitions of the number 3 are

$$3 = \begin{cases} 1\cdot 3, \;\; \nu = 1 \\ 3 \cdot 1, \;\; \nu = 1 \\ 1\cdot 1 + 2\cdot 1, \;\; \nu = 2 \quad . \end{cases} \tag{111}$$

Thus the exponential of (17) may be expanded, and we find

$$f_N^{(2n)} = \sum_{\pi\in\mathcal{P}(n)} \prod_{i=1}^{\nu(\pi)} \frac{1}{m_i!} \left(F_N^{(2n_i)}\right)^{m_i}, \tag{112}$$

where the sum is over the set of partitions $\mathcal{P}(n)$ of the number $n$. Thus $f_N^{(2n)}$ is the sum of all $2n$ dimensional integrals in (17). Explicitly

$$f_N^{(2n)} = \frac{(-1)^n}{(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^N \prod_{j=1}^{n} Q(z_{2j-1})Q(z_{2j-1}^{-1})P(z_{2j})P(z_{2j}^{-1}) \frac{1}{1-z_{2j-1}z_{2j}}$$
$$\sum_{\pi\in\mathcal{P}(n)} \prod_{k=1}^{\nu(\pi)} (-1)^{m_k} \frac{1}{m_k! n_k^{m_k}} \prod_{p=1}^{m_k} \prod_{q=1}^{n_k} \frac{1}{1 - z_{\sum_{r=1}^{k-1} 2m_r n_r + 2(p-1)n_k + 2q} \, z_{\sum_{r=1}^{k-1} 2m_r n_r + 2(p-1)n_k + (2q \oplus_{\pi,k} 1)}} \tag{113}$$

where

$$2q \oplus_{\pi,k} 1 := \begin{cases} 2q+1 & \text{if } q < n_k \\ 1 & \text{if } q = n_k. \end{cases} \tag{114}$$

We see that a partition $\pi$ divides the integrand into $\sum_{k=1}^{\nu(\pi)} m_k$ loops, and that there are $m_k$ loops of length $n_k$. As an illustration,

$$\begin{aligned} f_N^{(6)} = \quad & -\frac{1}{(2\pi)^6} \lim_{\epsilon\to 0} \prod_{i=1}^{6} \oint_{|z_i|=1-\epsilon} dz_i \prod_{j=1}^{3} Q(z_{2j-1}) Q(z_{2j-1}^{-1}) P(z_{2j}) P(z_{2j}^{-1}) \prod_{l=1}^{6} z_l^N \\ & \frac{1}{1-z_1 z_2} \frac{1}{1-z_3 z_4} \frac{1}{1-z_5 z_6} \\ & \Big( -\frac{1}{3!} \frac{1}{1-z_2 z_1} \frac{1}{1-z_4 z_3} \frac{1}{1-z_6 z_5} - \frac{1}{3} \frac{1}{1-z_2 z_3} \frac{1}{1-z_4 z_5} \frac{1}{1-z_6 z_1} \\ & + \frac{1}{2} \frac{1}{1-z_2 z_1} \frac{1}{1-z_4 z_5} \frac{1}{1-z_6 z_3} \Big) \end{aligned} \tag{115}$$

The first term in the bracket of the right hand side of (115) comes from $\pi_1 = \{(1,3)\}$, the second from $\pi_2 = \{(3,1)\}$ and the third from $\pi_3 = \{(1,1),\ (2,1)\}$. We would like to show that

$$\begin{aligned} f_N^{(2n)} = \quad & \frac{1}{n!(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i\ z_i^N \prod_{j=1}^{n} Q(z_{2j-1}) Q(z_{2j-1}^{-1}) P(z_{2j}) P(z_{2j}^{-1}) \frac{1}{1-z_{2j-1} z_{2j}} \\ & \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) \prod_{k=1}^{n} \frac{1}{1-z_{2k} z_{\sigma(2k-1)}}, \end{aligned} \tag{116}$$

where $S_n$ is the group of permutations of the $n$ elements $\{2i-1\}_{i=1}^n$. For instance,

$$S_3 = \{(1)(3)(5),\ (13)(5),\ (15)(3),\ (35)(1),\ (135),\ (153)\} \tag{117}$$

where the loop $(abc)$ means the permutation $a \to b \to c \to a$. We say that two permutations $\sigma_1$ and $\sigma_2$ in $S_n$ are equivalent if for every loop in $\sigma_1$ there is one and only one loop of equal length in $\sigma_2$. Then $\sigma_1$ and $\sigma_2$ will also have the same signature. We write the equivalence class of an element $\sigma$ as $[\sigma]$. We denote by $E_n$ the set of equivalence classes of $S_n$. As an example, we have

$$E_3 = \{[(1)(3)(5)],\ [(13)(5)],\ [(135)]\} \tag{118}$$

We will show that there is a bijection between $\mathcal{P}(n)$ and $E_n$. It is clear that $|\mathcal{P}(3)| = |E_3| = 3$. We will now prove the general case.

We will now calculate $|[\sigma]|$, the number of elements of the equivalence class of a permutation $\sigma$. We consider some $\sigma \in S_n$, and construct $[\sigma]$ as follows. We choose freely from $n$ elements, and divide them into $\sum_{i=1}^{\nu} m_i$ loops such that there are $m_i$ loops with $n_i$ elements, without distinguishing between loops with the same number of elements. There are

$$\frac{n!}{\prod_{i=1}^{\nu} (n_i!)^{m_i} m_i!} \tag{119}$$

ways of doing this. There are $(n_i - 1)!$ ways of ordering a loop of $n_i$ elements. Hence, the there are

$$|[\sigma]| = \frac{n! \prod_{i=1}^{\nu} ((n_i-1)!)^{m_i}}{\prod_{j=1}^{\nu} (n_j!)^{m_j} m_j!} = \frac{n!}{\prod_{i=1}^{\nu} n_i^{m_i} m_i!} \tag{120}$$

ways of choosing the elements. The signature of any element of the equivalence class $[\sigma]$ corresponding to $\pi$ is

$$\operatorname{sign}(\sigma) = (-1)^n \prod_{k=1}^{\nu(\pi)} (-1)^{m_k}. \tag{121}$$

If we identify every equivalence class with one of its representatives, then it follows from (113) that

$$f_N^{(2n)} = \frac{1}{n!(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i\ z_i^N \prod_{j=1}^{n} Q(z_{2j-1})Q(z_{2j-1}^{-1})P(z_{2j})P(z_{2j}^{-1})\ \frac{1}{1-z_{2j-1}z_{2j}}$$
$$\sum_{\sigma\in E_n} \mathrm{sign}(\sigma)\ |[\sigma]| \prod_{k=1}^{n} \frac{1}{1-z_{2k}z_{\sigma(2k-1)}}. \tag{122}$$

Now (116) follows. By symmetry of the odd variables, (116) can be rewritten as

$$f_N^{(2n)} = \frac{1}{(n!)^2(2\pi)^{2n}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n} \oint_{|z_i|=1-\epsilon} dz_i\ z_i^N \prod_{j=1}^{n} Q(z_{2j-1})Q(z_{2j-1}^{-1})P(z_{2j})P(z_{2j}^{-1})$$
$$\left(\sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{k=1}^{n} \frac{1}{1-z_{2k}z_{\sigma(2k-1)}}\right)^2. \tag{123}$$

Finally we note that the factor of the integrand of (123)

$$\sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{k=1}^{n} \frac{1}{1-z_{2k}z_{\sigma(2k-1)}} \tag{124}$$

is zero if for any $i \neq j$, $z_{2i} = z_{2j}$ or $z_{2i-1} = z_{2j-1}$. Therefore

$$\sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{k=1}^{n} \frac{1}{1-z_{2k}z_{\sigma(2k-1)}} = A_n \prod_{k=1}^{n}\prod_{l=1}^{n} \frac{1}{(1-z_{2k-1}z_{2l})} \prod_{1\le p<q\le n} (z_{2p-1}-z_{2q-1})(z_{2p}-z_{2q}). \tag{125}$$

By letting $z_{2n} = z_{2n-1} = 0$ we find that

$$A_n = A_{n-1}. \tag{126}$$

Since $A_1 = 1$, it follows that $A_n = 1$ for all $n$. Hence we obtain the desired result (21).

## VI. THE FORM FACTOR EXPANSION FOR $T > T_c$

Above $T_c$, $D_N^{(+)}$ has a form factor expansion given by (31), where

$$f_N^{(2n+1)} = \sum_{k=0}^{n} G_N^{(2k+1)} \hat{f}_{N+1}^{(2n-2k)} \tag{127}$$

and $\hat{f}_N^{(2n)}$ is given by (116) but with $P$ and $Q$ replaced by $\hat{P}$ and $\hat{Q}$. $G_N^{(2n+1)}$ is given by (29). Hence it follows from (116), (127) and (29) that

$$f_N^{(2n+1)} = -\frac{i}{(2\pi)^{2n+1}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n+1} \oint_{|z_i|=1-\epsilon} dz_i\ z_i^{N+1} \prod_{l=1}^{n+1} \hat{P}(z_{2l-1})\hat{P}(z_{2l-1}^{-1}) \prod_{m=1}^{n} \hat{Q}(z_{2m})\hat{Q}(z_{2m}^{-1}) \frac{1}{z_{2n+1}} \prod_{p=1}^{n} \frac{1}{1-z_{2p-1}z_{2p}}$$
$$\sum_{k=0}^{n} (-1)^k \frac{1}{(n-k)!} \frac{1}{z_{2n-2k+1}} \sum_{\sigma\in S_{n-k}} \mathrm{sign}(\sigma) \prod_{q=1}^{n-k} \frac{1}{1-z_{2q-1}z_{\sigma(2q)}} \prod_{s=n-k+1}^{n} \frac{1}{1-z_{2s}z_{2s+1}}. \tag{128}$$

As an example,

$$f_N^{(5)} = -\frac{i}{(2\pi)^5} \lim_{\epsilon\to 0} \prod_{i=1}^{5} \oint_{|z_i|=1-\epsilon} dz_i\ z_i^{N+1} \prod_{l=1}^{3} \hat{P}(z_{2l-1})\hat{P}(z_{2l-1}^{-1}) \prod_{m=1}^{2} \hat{Q}(z_{2m})\hat{Q}(z_{2m}^{-1}) \frac{1}{z_5} \frac{1}{1-z_1z_2} \frac{1}{1-z_3z_4}$$
$$\left(\frac{1}{2}\frac{1}{z_5}\left(\frac{1}{1-z_1z_2}\frac{1}{1-z_3z_4} - \frac{1}{1-z_1z_4}\frac{1}{1-z_2z_3}\right) - \frac{1}{z_3}\frac{1}{1-z_1z_2}\frac{1}{1-z_4z_5} + \frac{1}{z_1}\frac{1}{1-z_2z_3}\frac{1}{1-z_4z_5}\right). \tag{129}$$

Let $(i_1^{(k)},...,i_n^{(k)}) := (1,...,n-k,n-k+2,...,n+1)$. It follows by symmetry that (128) can be rewritten as

$$\begin{aligned} f_N^{(2n+1)} = & -\frac{i}{(2\pi)^{2n+1}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n+1} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^{N+1} \prod_{l=1}^{n+1} \widehat{P}(z_{2l-1})\widehat{P}(z_{2l-1}^{-1}) \prod_{m=1}^{n} \widehat{Q}(z_{2m})\widehat{Q}(z_{2m}^{-1}) \\ & \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r \frac{1}{z_{2n-2r+1}} \prod_{p=1}^{n} \frac{1}{1 - z_{2i_p^{(r)}-1} z_{2i_p^{(r)}}} \\ & \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \frac{1}{z_{2n-2k+1}} \sum_{\sigma\in S_n} \mathrm{sign}(\sigma) \prod_{q=1}^{n} \frac{1}{1 - z_{2i_q^{(k)}-1} z_{\sigma(2i_q^{(k)})}}. \end{aligned} \tag{130}$$

In particular

$$\begin{aligned} f_N^{(5)} = & -\frac{i}{(2\pi)^5} \lim_{\epsilon\to 0} \prod_{i=1}^{5} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^{N+1} \prod_{l=1}^{3} \widehat{P}(z_{2l-1})\widehat{P}(z_{2l-1}^{-1}) \prod_{m=1}^{2} \widehat{Q}(z_{2m})\widehat{Q}(z_{2m}^{-1}) \\ & \frac{1}{3}\left( \frac{1}{z_5}\frac{1}{1-z_1z_2}\frac{1}{1-z_3z_4} - \frac{1}{z_3}\frac{1}{1-z_1z_2}\frac{1}{1-z_4z_5} + \frac{1}{z_1}\frac{1}{1-z_2z_3}\frac{1}{1-z_4z_5} \right) \\ & \left\{ \frac{1}{2}\frac{1}{z_5}\left( \frac{1}{1-z_1z_2}\frac{1}{1-z_3z_4} - \frac{1}{1-z_1z_4}\frac{1}{1-z_2z_3} \right) \right. \\ & -\frac{1}{2}\frac{1}{z_3}\left( \frac{1}{1-z_1z_2}\frac{1}{1-z_4z_5} - \frac{1}{1-z_1z_4}\frac{1}{1-z_2z_5} \right) \\ & \left. +\frac{1}{2}\frac{1}{z_1}\left( \frac{1}{1-z_2z_3}\frac{1}{1-z_4z_5} - \frac{1}{1-z_3z_4}\frac{1}{1-z_2z_5} \right) \right\}. \end{aligned} \tag{131}$$

Since all permutations of the even elements are present in the sum $\sum_{k=0}^{n}$, symmetry allows the permutation of all even elements in the sum $\sum_{r=0}^{n}$. But the sum $\sum_{k=0}^{n}\sum_{\sigma\in S_n}$ may be rewritten as the sum $\sum_{\sigma\in S_{n+1}}$ of permutations of the odd elements. Therefore

$$\begin{aligned} f_N^{(2n+1)} = -\frac{i}{(2\pi)^{2n+1}} \lim_{\epsilon\to 0} \prod_{i=1}^{2n+1} \oint_{|z_i|=1-\epsilon} dz_i \; z_i^{N+1} \prod_{l=1}^{n+1} \widehat{P}(z_l)\widehat{P}(z_l^{-1}) \prod_{m=1}^{n} \widehat{Q}(z_m)\widehat{Q}(z_m^{-1}) \\ \frac{1}{n!(n+1)!} \left( \sum_{\sigma\in S_{n+1}} \mathrm{sign}(\sigma) \frac{1}{z_{\sigma(2n+1)}} \prod_{q=1}^{n} \frac{1}{1 - z_{\sigma(2q-1)} z_{2q}} \right)^2. \end{aligned} \tag{132}$$

An argument similar to the one given in section V shows that

$$\begin{aligned} \sum_{\sigma\in S_{n+1}} \mathrm{sign}(\sigma) \frac{1}{z_{\sigma(2n+1)}} \prod_{q=1}^{n} \frac{1}{1 - z_{\sigma(2q-1)} z_{2q}} = & \prod_{j=1}^{n+1} \frac{1}{z_{2j-1}} \prod_{k=1}^{n} \frac{1}{1 - z_{2j-1} z_{2k}} \\ & \prod_{1\le l<m\le n+1} (z_{2l-1} - z_{2m-1}) \prod_{1\le p<q\le n} (z_{2p} - z_{2q}). \end{aligned} \tag{133}$$

Thus $f_N^{(2n+1)}$ is given by (32) as desired.

## VII. DISCUSSION

The exponential and the form factor representations derived in this paper for $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ are considerably simpler that the corresponding representations which may be found in [10]-[15]. The representations of this paper must of course be equal to the corresponding results of [10]-[15] but as mentioned in the introduction even the equality of the form of $F_N^{(2)}$ found by Wu [7] with the form found by Cheng and Wu [6] has not been directly demonstrated in the literature. The form factor representations for $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ proven here are in close correspondence with formulas given by Jimbo and Miwa [21] in their proof of the Painleve VI representation of the diagonal Ising correlations. The connection which the form factor representations of this paper have with the PVI equation of [21]

have been extensively investigated in [17]. However, the representations of this paper are valid also for $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ and, as noted in the introduction, for much more general case which suggests that there are generalizations of [21] which have not yet been uncovered. In particular the relation of $\langle\sigma_{0,0}\sigma_{0,N}\rangle$ to isomonodromic deformation theory remains to be investigated.

**Acknowledgments**

We would like to thank Prof. J.-M. Maillard and Prof. N. Zenine for many insightful discussions and Prof. J.-M. Maillet for discussions and hospitality at Ecole Normale Superieur Lyon where part of this work was performed. This work is supported in part by NSF grant DMR-0302758.

[1] B. Kaufman and L. Onsager, Crystal statistics. III. Short-range order in a binary Ising model, Phys. Rev. 76 (1949) 1244.
[2] E. W. Montroll, R. B. Potts and J. C. Ward, Correlations and spontaneous magnetization of the two dimensional Ising model, J. Math. Phys. 4 (1963) 308.
[3] J. Stephenson, Ising model spin correlations on the triangular latice, J. Math. Phys. 5 (1964) 1009.
[4] K. Yamada, Pair Correlation Function in the Ising Square Lattice - Determinental Form, Prog. Theor. Phys. 72 (1984) 922.
[5] K. Yamada, Pair Correlation Function in the Ising Square Lattice - Generalized Wronskian Form, Prog. Theor. Phys. 76 (1986) 602.
[6] H. Cheng and T.T. Wu, Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. III, Phys. Rev. 162 (1967) 436.
[7] T. T. Wu, Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model, Phys. Rev. 149 (1966) 380.
[8] G. Szegö, Commun. Seminair, Math. Univ. Lund, suppl. dedié a Marcel Riesz, 228 (1952).
[9] V. Grenander and G. Szegö, Toeplitz Forms and their Applications, University of California Press, Berkeley, 1958.
[10] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region, Phys. Rev. B 13 (1976) 316.
[11] C. R. Nappi, On the Scaling Limit of the Ising Model, Nuovo Cimento A 44 (1978) 392.
[12] J. Palmer and C. Tracy, Two-Dimensional Ising Correlations: Convergence of the Scaling Limit, Adv. Appl. Math. 2 (1981) 329.
[13] K. Yamada, On the Spin-Spin Correlation Function in the Ising Square Lattice and the Zero Field Susceptibility, Prog. Theor. Phys. 71 (1984) 1416.
[14] B. Nickel, On the singularity of the 2D Ising model susceptibility, J. Phys. A 32 (1999) 3889.
[15] B. Nickel, Addendum to “On the singularity of the 2D Ising model susceptibility”, J. Phys. A 33 (2000) 1693.
[16] B. M. McCoy and T. T. Wu, *The Two Dimensional Ising Model* (Harvard Univ. Press, Cambridge Mass. 1973) chapters VIII and XI.
[17] S. Boukraa, S. Hassani, J.-M. Maillard, B. M. McCoy, W. P. Orrick and N. Zenine, Holonomy of Ising model form factors, J. Phys. A 40 (2007) in press math-ph/0609074.
[18] E. Lieb, T. Schultz and D. Mattis, Two soluable models of an antiferromagnetic chain, Ann. Phys. 16 (1961) 407.
[19] B.M. McCoy, Spin correlation functions of the XY model, Phys. Rev. 173 (1968) 531.
[20] E. Barouch and B. McCoy, Statistical mechanics of the XY model II. Spin correlation functions, Phys. Rev, A3 (1971) 786
[21] M. Jimbo and T. Miwa, Studies on Holonomic Quantum Fields XVII, Proc. Japan Acad. 56A (1980) 405-410;57A (1981) 347.